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NORMAL STRUCTURE AND WEAKLY NORMAL STRUCTURE OF ORLICZ SEQUENCE SPACES

BY

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ABSTRACT. For a convex Orlicz function $\varphi\colon \mathbf{R}_+ \to \mathbf{R}_+ \cup \{\infty\}$ and the associated Orlicz sequence space l_φ, we consider the following five properties:

(1) l_φ has a subspace isometric to l_1.

(2) l_φ is Schur.

(3) l_φ has normal structure.

(4) Every weakly compact subset of l_φ has normal structure.

(5) Every bounded sequence in l_φ has a subsequence (x_n) which is pointwise and almost convergent to $x \in l_\varphi$, i.e., $\limsup_{n\to\infty}\|x_n - x\|_\varphi < \liminf_{n\to\infty}\|x_n - y\|_\varphi$ for all $y \neq x$.

Our results are:

(1) $\Leftrightarrow$ φ is either linear at 0 ($\varphi(s)/s = c > 0$, $0 < s \leqslant t$) or does not satisfy the Δ_2-condition at 0.

(2) $\Leftrightarrow$ l_φ is isomorphic to $l_1 \Leftrightarrow \varphi'(0) = \lim_{t\to 0}\varphi(t)/t > 0$.

(3) $\Leftrightarrow$ φ satisfies the Δ_2-condition at 0, φ is not linear at 0 and $C(\varphi) = \sup\{\varphi(t) < 1\} > \frac{1}{2}$.

(4) $\Leftrightarrow$ φ satisfies the Δ_2-condition at 0 and $C(\varphi) > \frac{1}{2}$ or $\varphi'(0) > 0$.

(5) $\Leftrightarrow$ φ satisfies the Δ_2-condition at 0 and $C(\varphi) = 1$.

The last equivalence contains a result of Lami-Dozo [10].

1. Preliminaries. Let X be a normed space and A be a nonvoid subset of X. The set A is called *diametral* if A is bounded and if

$$\sup\{\|x - y\| \mid y \in A\} = d \quad \text{for all } x \in A \text{ and some } d > 0,$$

and A is said to have *normal structure* (cf. [2]) if A has no convex diametral subset.

The normed space X is said to have *weakly normal structure* if every weakly compact subset of X has normal structure.

Simple examples of sets with normal structure are the subsets of uniformly convex spaces and the compact subsets of any normed space. More examples and information may be found in [12].

Normal structure is one of the fundamental tools in fixed point theory of nonexpansive mappings. We mention only the fixed point theorem of Browder [3], Göhde [6] and Kirk [8] which states that every nonexpansive (i.e., having Lipschitz-constant 1) selfmap of A has a fixed point provided A has normal structure and is weakly compact and convex.

For a given sequence $(x_n)_{n\in\mathbf{N}}$ in X, we associate two functionals

$$\Lambda^*(x) := \limsup_{n\to\infty}\|x_n - x\|, \qquad \Lambda_*(x) := \liminf_{n\to\infty}\|x_n - x\|$$

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on X (cf. [**10**]). If $(x_n)_{n\in\mathbf{N}}$ is bounded, $(x_n)_{n\in\mathbf{N}}$ is called *limit-constant* (*limit-affine*) if the two functionals Λ^* and Λ_* coincide on the convex hull of $\{x_n | n \in \mathbf{N}\}$ and are constant (affine) there (cf. [**12**]). In this case, we write $\Lambda = \Lambda^* = \Lambda_*$.

PROPOSITION 1 [**12**]. *The normed space X has (weakly) normal structure if and only if X contains no (weakly convergent) limit-constant nonconstant sequence.*

The normed space X is said to have the (*weak*) *sum-property* (cf. [**12**]) if it contains no (weakly convergent) nonconstant limit-affine sequence $(x_n)_{n\in\mathbf{N}}$ for which $(\Lambda(x_n))_{n\in\mathbf{N}}$ is nondecreasing. The sum-property is introduced in [**12**] to solve (at least partially) the problem whether normal structure is preserved under finite direct sums. Here, a property (P) is said to be *preserved under finite direct sums* if, given a finite dimensional normed space Z having a basis $(e_i)_{i\leqslant N}$ with

$$\left\|\sum_{i=1}^{N} \zeta_i e_i\right\|_Z \leqslant \left\|\sum_{i=1}^{N} \zeta_i' e_i\right\|_Z \quad \text{whenever } \zeta_i' \geqslant \zeta_i \geqslant 0 \text{ for all } i \leqslant N$$

and given normed space X_i, $i \leqslant N$, with (P), then $\prod_{i=1}^{N} X_i$ endowed with the norm $\|(x_i)_{i\leqslant N}\| = \|\sum_{i=1}^{N} \|x_i\|_{X_i} e_i\|_Z$ has (P).

In [**12**], it is proved that the sum-property is preserved under finite direct sums. So, since the sum-property obviously implies normal structure, it is desirable to find spaces having the sum-property or even to show that normal structure is equivalent to the sum-property. The latter is an open problem and is shown to hold for a very large class of spaces (cf. [**12**]).

We will see that, among the Orlicz sequence spaces, weakly normal structure is equivalent to the weak sum-property and normal structure is equivalent to the sum-property.

The idea of the following sufficient condition for weakly normal structure of normed spaces with a basis is due to Gossez and Lami-Dozo [**7**].

PROPOSITION 2 [**12**]. *Let X have a basis $(e_i)_{i\in\mathbf{N}}$ which satisfies the condition*

(GLD). *There are $\varepsilon < 1$ and $r > 0$ such that* $\|x\| = \|\sum_{i=1}^{\infty} x(i)e_i\| \geqslant 1 + r$ *whenever* $\|\sum_{i=1}^{j} x(i)e_i\| = 1$ *and* $\|\sum_{i=j+1}^{\infty} x(i)e_i\| \geqslant \varepsilon$ *for some* $j \in \mathbf{N}$.

Then, X contains no weakly convergent limit-affine sequence with $\inf\{\Lambda(x_n) | n \in \mathbf{N}\} > 0$. *Consequently, X has weakly normal structure, it even has the weak sum-property.*

A sequence $(x_n)_{n\in\mathbf{N}}$ in X is said to be *almost convergent* to some $x \in X$ if $\Lambda^*(x) < \Lambda_*(y)$ for all $y \neq x$ (cf. [**4, 9**]). This notion is useful in fixed point theory of nonexpansive mappings: If T is a nonexpansive mapping and $(x_n)_{n\in\mathbf{N}}$ is a sequence with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ which is almost convergent to some x, then x is a fixed point of T.

2. Orlicz sequence spaces. Let φ be a convex *Orlicz function*, i.e., $\varphi\colon \mathbf{R}_+ \to \overline{\mathbf{R}}_+ = \mathbf{R}_+ \cup \{+\infty\}$ is nondecreasing, nonconstant, convex and continuous at 0 with $\varphi(0) = 0$. We set

$$D(\varphi) = \{t > 0 | \sup\{\varphi(s) | 0 < s < t\} < \infty\}.$$

Every Orlicz function φ is continuous on the interior of $D(\varphi)$ and possesses a right derivative $\varphi'(t) = \lim_{h \downarrow 0}\{(\varphi(t+h) - \varphi(t))/h\}$ there.

Throughout the paper, let $\mathbf{K}$ be the scalar field $\mathbf{R}$ or $\mathbf{C}$. We define the functional $S_\varphi: \mathbf{K}^{\mathbf{N}} \to \overline{\mathbf{R}}_+$ by

$$S_\varphi(x) = \sum_{i=1}^{\infty} \varphi(|x(i)|).$$

The *Orlicz sequence space* $l_\varphi = l_\varphi(\mathbf{K})$ is defined by

$$l_\varphi = \left\{ x: \mathbf{N} \to \mathbf{K} | S_\varphi(c^{-1}x) < \infty \text{ for some } c > 0 \right\}.$$

Endowed with the norm

$$\|x\|_\varphi = \inf\left\{ c > 0 | S_\varphi(c^{-1}x) \leqslant 1 \right\}$$

l_φ is a real or complex Banach space according to $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. As usual, we write $l_p = l_{\varphi_p}$ and $\| \, \|_p = \| \, \|_{\varphi_p}$ for $\varphi_p(t) = t^p$, $1 \leqslant p < \infty$, and

$$\varphi_p(t) = \begin{Bmatrix} 0, & t \leqslant 1 \\ \infty, & t > 1 \end{Bmatrix}, \qquad p = \infty.$$

The unit vectors in l_φ are denoted by $(e_i)_{i \in \mathbf{N}}$, where

$$e_i(j) = \begin{Bmatrix} 0, & i \neq j \\ 1, & i = j \end{Bmatrix}.$$

Whether $(e_i)_{i \in \mathbf{N}}$ is a basis of l_φ depends strongly on the so-called Δ_2-condition. The Orlicz function φ is said to satisfy the *Δ_α-condition at* 0, $\alpha > 1$, if it is *nondegenerate* —i.e., if $\varphi(t) > 0$ for all $t > 0$—and if there is $t > 0$ with $\alpha t \in D(\varphi)$ and

$$\beta(\alpha, t) := \sup\left\{ \varphi(\alpha s)/\varphi(s) | 0 < s < t \right\} < \infty.$$

Obviously, $\beta(\alpha, t) < \infty$ whenever $\alpha t \in D(\varphi)$ and φ satisfies the Δ_α-condition at 0. It is important in our results that $l_\varphi \subset c_0$ if φ is nondegenerate. The following fact is well known (cf. [**13**]):

PROPOSITION 3. *The following are equivalent.*

(a) φ *satisifes the* Δ_α*-condition at* 0 *for all* $\alpha > 1$.

(b) φ *satisifes the* Δ_α*-condition at* 0 *for all some* $\alpha > 1$.

(c) φ *satisifes the* Δ_2*-condition at* 0.

(d) $\limsup_{t \downarrow 0} t\varphi'(t)/\varphi(t) < \infty$.

(e) $(e_i)_{i \in \mathbf{N}}$ *is a boundedly complete basis of* l_φ.

(f) l_φ *has no subspace isomorphic to* l_∞.

Although Lindenstrauss and Tzafriri only deal with real valued φ in [**13**], the proof of Proposition 3 also works in our case.

We need some elementary properties of Orlicz functions and Orlicz sequence spaces.

PROPOSITION 4. (i) *If* φ *satisfies the* Δ_2*-condition at* 0, *then*

$$\lim_{\alpha \downarrow 1} \beta(\alpha, \alpha^{-1}t) = 1 \quad \textit{for all } t \in D(\varphi).$$

(ii) *If* φ *does not satisfy the* Δ_2*-condition at* 0, *then, for all* $t > 0$ *and* $\beta < \infty$, *there is* $s \in (0, t]$ *with*

$$\infty > \varphi(\alpha s) \geqslant (\alpha - 1)\beta\varphi(s) \quad \textit{for all } \alpha \in (1,2].$$

PROOF. (i) Assume that φ satisfies the Δ_2-condition at 0. Let $t \in D(\varphi)$ and $\beta > 1$ be given. We have to find $\alpha > 1$ such that $\beta(\alpha, \alpha^{-1}t) \leqslant \beta$. Since $t \in D(\varphi)$, there is $\tilde{\alpha} > 1$ such that $\varphi(s) \leqslant \beta\varphi(\tilde{\alpha}^{-1}t)$ for all $s < t$. Set $\gamma = \beta(\tilde{\alpha}, \tilde{\alpha}^{-1}t)$ and choose $\alpha \in (1, \tilde{\alpha}]$ with $\gamma(\alpha - 1) + \tilde{\alpha} - \alpha \leqslant \beta(\tilde{\alpha} - 1)$. For $s \in [\tilde{\alpha}^{-1}t, \alpha^{-1}t)$, we have $\varphi(\alpha s) \leqslant \beta\varphi(\tilde{\alpha}^{-1}t) \leqslant \beta\varphi(s)$. Convexity of φ implies for $s \in (0, \tilde{\alpha}^{-1}t)$:

$$\varphi(\alpha s) \leqslant \frac{\alpha - 1}{\tilde{\alpha} - 1}\varphi(\tilde{\alpha}s) + \frac{\tilde{\alpha} - \alpha}{\tilde{\alpha} - 1}\varphi(s) \leqslant \left(\gamma\frac{\alpha - 1}{\tilde{\alpha} - 1} + \frac{\tilde{\alpha} - \alpha}{\tilde{\alpha} - 1}\right)\varphi(s) \leqslant \beta\varphi(s).$$

(ii) Assume that φ does not satisfy the Δ_2-condition at 0. Let $t > 0$ and $\beta < \infty$ be given. Choose $\tilde{t} \leqslant t$ with $\varphi(2\tilde{t}) < \infty$. By Proposition 3(d), there is $s \in (0, \tilde{t}]$ with $s\varphi'(s)/\varphi(s) \geqslant \beta$. Fix $\alpha \in (1,2]$. Since $(\varphi(\alpha s) - \varphi(s))/((\alpha - 1)s) \geqslant \varphi'(s)$ we finally obtain $\varphi(\alpha s)/\varphi(s) \geqslant 1 + (\alpha - 1)s\varphi'(s)/\varphi(s) \geqslant (\alpha - 1)\beta$. □

The behaviour of φ on $\varphi^{-1}([0,1])$ is important for the norm of l_φ. We define

$$t(\varphi) = \sup\{\, t|\varphi(t) \leqslant 1\}, \qquad C(\varphi) = \sup\{\, \varphi(t)|t < t(\varphi)\}.$$

PROPOSITION 5. *If* φ *satisfies the* Δ_2*-condition at* 0, *then* $\lim_{\lambda \uparrow 1} S_\varphi(\lambda x) \geqslant C(\varphi)$ *uniformly on* $\{x \in l_\varphi |\, \|x\|_\varphi = 1\}$.

PROOF. Assume the contrary. Then there is $C < C(\varphi)$ such that, for all $\lambda < 1$, there is x_λ with $\|x_\lambda\|_\varphi = 1$ and $S_\varphi(\lambda x_\lambda) < C$. Since $C < C(\varphi)$, there is $t > 0$ with $\varphi(t) = C$ and $\tilde{\alpha} > 1$ such that $\tilde{\alpha}t \in D(\varphi)$. Choose $\alpha \in (1, \tilde{\alpha}]$ with $\beta(\alpha, t) \leqslant C^{-1}$ and set $\lambda = \alpha^{-1/2}$. Since $S_\varphi(\lambda x_\lambda) < C$, we have $\lambda\|x_\lambda\|_\infty < t$. So, $S_\varphi(\sqrt{\alpha}\, x_\lambda) = S_\varphi(\alpha\lambda x_\lambda) \leqslant C^{-1}S_\varphi(\lambda x_\lambda) < 1$ which contradicts $\|x_\lambda\|_\varphi = 1$. □

The following condition (G_ε) defined for $\varepsilon \in (0,1)$ is closely related to the (GLD)-condition and to another condition (cf. [**10**]) which implies that any bounded sequence in l_φ has a subsequence which is pointwise and almost convergent.

(G_ε) There exists $r > 0$ such that $\|x + y\|_\varphi \geqslant 1 + r$ whenever $\|x\|_\varphi = 1$, $\|y\|_\varphi \geqslant \varepsilon$ and x, y have disjoint supports.

The validity of (G_ε) for appropriate ε depends on the number $C(\varphi)$:

PROPOSITION 6. *Let* φ *satisfy the* Δ_2*-condition at* 0.

(a) *Condition* (G_ε) *holds for some* $\varepsilon \in (0,1)$ *if and only if* $C(\varphi) > \frac{1}{2}$.

(b) *Condition* (G_ε) *holds for all* $\varepsilon \in (0,1)$ *if and only if* $C(\varphi) = 1$.

PROOF. Assume $C = \max\{\frac{1}{2}, C(\varphi)\} < 1$. Set $\varepsilon = C^{-1} - 1$. Then $\varepsilon = 1$ if $C = \frac{1}{2}$. We have $\|x\|_\varphi = t(\varphi)^{-1}$ whenever $\|x\|_1 \leqslant C^{-1}$ and $|x(j)| = 1$ for some $j \in \mathbf{N}$. Indeed, $S_\varphi(tx) \geqslant \varphi(t|x(j)|) = \varphi(t) > 1$ if $t > t(\varphi)$, and, since $\sum_{i \neq j}|x(i)| \leqslant \varepsilon \leqslant 1$, we have $S_\varphi(tx) \leqslant \varphi(t) + \varepsilon\varphi(t) = C^{-1}\varphi(t) < 1$ if $t < t(\varphi)$. So $\|t(\varphi)e_1\|_\varphi = 1$, $\|\varepsilon t(\varphi)e_2\|_\varphi = \varepsilon$ and $\|t(\varphi)e_1 + \varepsilon t(\varphi)e_2\|_\varphi = 1$. This proves sufficiency in (a) and (b).

Assume $C(\varphi) > \frac{1}{2}$ in case (a) and $C(\varphi) = 1$ in case (b). In case (b), let $\varepsilon > 0$ be given. For necessity in (a) and (b), it suffices to find $\lambda < 1$ such that $S_\varphi(\lambda x) + S_\varphi(\lambda y) > 1$ whenever $\|x\|_\varphi = 1$ and $\|y\|_\varphi = \varepsilon$, where, in case (a), $\varepsilon \in (0,1)$ has to be chosen in an appropriate manner.

In case (a), we choose $\lambda < 1$ and $C > \frac{1}{2}$ such that $S_\varphi(\lambda z) > C$ whenever $\|z\|_\varphi = 1$, and pick $\varepsilon \in (0,1)$ with $\gamma = \beta(\varepsilon^{-1}, \varepsilon t(\varphi)) < 2C$; we have $C(1 + \gamma^{-1}) > 1$.

In case (b), we put $\gamma = \beta(\varepsilon^{-1}, \varepsilon t(\varphi))$, pick $C < 1$ with $C(1 + \gamma^{-1}) > 1$ and choose $\lambda < 1$ such that $S_\varphi(\lambda z) > C$ whenever $\|z\|_\varphi = 1$.

Let x, y be given with $\|x\|_\varphi = 1$, $\|y\|_\varphi = \varepsilon$. Since $S_\varphi(\lambda \varepsilon^{-1} y) < 1$, we have $\lambda \|y\|_\infty < \varepsilon t(\varphi)$, so, $S_\varphi(\lambda\varepsilon^{-1}y) \leqslant \gamma S_\varphi(\lambda y)$. Hence $S_\varphi(\lambda x) + S_\varphi(\lambda y) > C + \gamma^{-1} S_\varphi(\lambda \varepsilon^{-1} y) \geqslant C(1 + \gamma^{-1}) > 1$. □

REMARK 1. In the first part of the proof we have shown in fact that $(e_n)_{n\in\mathbf{N}}$ is limit-constant if $C(\varphi) \leqslant \frac{1}{2}$, and that no subsequence of $(e_n)_{n\in\mathbf{N}}$ is almost convergent to its pointwise lilmit 0 if $C(\varphi) < 1$. Moreover, in this part of the proof, we did not use the Δ_2-condition.

The Orliz function φ is said to be *linear at* 0 if there are $t > 0$ and $c > 0$ such that $\varphi(s) = cs$ for all $s \leqslant t$.

PROPOSITION 7. *If φ is linear at* 0, *then l_φ has a subspace isometric to l_1.*

PROOF. Assume that $\varphi(s) = cs$, $s \leqslant t$, with $t > 0$, $c > 0$. Choose $m \in \mathbf{N}$ and $s \leqslant t$ with $mcs = 1$. Set $x_n = s\sum_{i=1}^m e_{mn+i}$. Fix any finite sequence $(a_k)_{k\leqslant n}$ of scalars and set $a = \sum_{k=1}^n |a_k|$. Then $S_\varphi(a^{-1}\sum_{k=1}^n a_k x_k) = \sum_{k=1}^n mca^{-1}|a_k|s = 1$, i.e., $\|\sum_{k=1}^n a_k x_k\|_\varphi = a = \sum_{k=1}^n |a_k|$. □

3. Main results. All results in this section are valid for $\mathbf{K} = \mathbf{R}$ as well as for $\mathbf{K} = \mathbf{C}$. Only in the proof of Theorem 1(d) do we have to distinguish between the cases $\mathbf{K} = \mathbf{R}$ and $\mathbf{K} = \mathbf{C}$.

Some of the proofs are very technical and involve a lot of estimations. So, we give these proofs in the last three sections.

We first characterize those Orlicz sequence spaces which contain an isometric copy of l_1.

THEOREM 1. *The Orlicz sequence space l_φ has a subspace isometric to l_1 if and only if φ does not satisfy the Δ_2-condition at* 0 *or φ is linear at* 0.

PROOF. See §4. □

It is well known that l_∞ contains an isometric copy of l_1. Theorem 1 tells us that l_φ contains an isometric copy of l_1 whenever l_φ has a subspace isomorphic—not necessarily isometric—to l_∞. This result depends on the special structure of Orlicz sequence spaces because there are normed spaces isomorphic to l_∞ which have normal structure (cf. [**12**]) and, consequently, cannot have a subspace isometric to l_1. On the other hand, if a normed space has a subspace isomorphic to l_∞, then, for all $\varepsilon > 0$, it has a subspace whose Banach-Mazur distance from l_1 is not bigger than $1 + \varepsilon$. Theorem 1 shows that, among Orlicz sequence spaces, this holds even for $\varepsilon = 0$.

We now show that the only Orlicz sequence spaces with the Schur property—weakly convergent sequences are norm convergent—are those isomorphic to l_1.

THEOREM 2. *The following are equivalent.*

(a) l_φ *is isomorphic to* l_1.

(b) l_φ *is Schur.*

(c) l_φ *has weakly normal structure with respect to every equivalent norm.*

(d) $(e_n)_{n\in\mathbf{N}}$ *does not converge weakly to* 0.

(e) $\varphi'(0) > 0$.

PROOF. (a) $\Rightarrow$ (b): l_1 is Schur. (b) $\Leftrightarrow$ (c): cf. [**11**]. (b) $\Rightarrow$ (d): Trivial. (d) $\Rightarrow$ (e): Assume $\varphi'(0) = 0$. Choose a real valued Orlicz function $\tilde{\varphi}$ with $\lim_{t\to\infty}\tilde{\varphi}'(t) = \infty$ such that $\varphi = \tilde{\varphi}$ in some neighbourhood of 0. Then $\tilde{\varphi}'(0) = 0$ and the conjugate Orlicz function $\tilde{\varphi}^*(u) = \sup_{t\geqslant 0}\{tu - \tilde{\varphi}(t)\}$ is nondegenerate, i.e. $l_{\tilde{\varphi}^*} \subset c_0$. Since the identity is an isomorphism of l_φ onto $l_{\tilde{\varphi}}$ and $l_\varphi^* \approx l_{\tilde{\varphi}^*}$ (cf. [**13**]), l_φ^* can be identified with $l_{\tilde{\varphi}^*}$ endowed with some equivalent norm. So, $\langle x^*, e_n\rangle = x^*(n) \to 0$, $n \to \infty$, for all $x^* \in l_\varphi^*$. (e) $\Rightarrow$ (a): If $\varphi'(0) = c > 0$, then $\varphi(t) \geqslant ct$ for all $t > 0$. So, the identity is an isomorphism of l_φ onto l_1 (cf. [**13**]). $\square$

In the next theorem, we give a characterization of Orlicz sequence spaces having normal structure.

THEOREM 3. *The following are equivalent.*

(a) l_φ *has normal structure.*

(b) l_φ *has the sum-property.*

(c) φ *satisfies the* Δ_2*-condition at* 0, φ *is not linear at* 0 *and* $C(\varphi) > \frac{1}{2}$.

(d) l_φ *has no subspace isometric to* l_1 *and* $C(\varphi) > \frac{1}{2}$.

PROOF. (b) $\Rightarrow$ (a): Trivial. (c) $\Leftrightarrow$ (d): Theorem 1. (a) $\Rightarrow$ (d): If l_φ has a subspace isometric to l_1 then l_φ cannot have normal structure. If $C(\varphi) \leqslant \frac{1}{2}$, then, according to Remark 1, $(e_n)_{n\in\mathbf{N}}$ is limit-constant; thus, l_φ does not have normal structure. (c) $\Rightarrow$ (b): See §5. $\square$

COROLLARY 1. *If* φ *is real valued or if* $\frac{1}{2} < \varphi(t) < \infty$ *for at least one* t, *then* l_φ *has normal structure if and only if* l_φ *has no subspace isometric to* l_1.

COROLLARY 2. *A reflexive Orlicz sequence space* l_φ *has normal structure if and only if* $C(\varphi) > \frac{1}{2}$.

EXAMPLE. Let φ be an arbitrary real valued Orlicz function. For $\lambda > 0$ set $\varphi|_\lambda(t) = \varphi(t)$ if $t \leqslant \lambda^{-1}$ and $\varphi|_\lambda(t) = \infty$ if $t > \lambda^{-1}$. Then the identity is an isomorphism of l_φ onto $l_{\varphi|_\lambda}$ and $\|x\|_{\varphi|_\lambda} = \max\{\|x\|_\varphi, \lambda\|x\|_\infty\}$. Since φ and $\varphi|_\lambda$ coincide in a neighborhood of 0 and $C(\varphi|_\lambda) = \min\{\varphi(\lambda^{-1}), 1\}$, Theorem 3 yields

COROLLARY 3. $l_{\varphi|_\lambda}$ *has normal structure if and only if* l_φ *has normal structure and* $\varphi(\lambda^{-1}) > \frac{1}{2}$.

Let $l_{p,\lambda}$ be the space l_p, $1 \leqslant p < \infty$, with norm $\|x\|_{p,\lambda} = \max\{\|x\|_p, \lambda\|x\|_\infty\}$. Since $l_{p,\lambda} = l_{\varphi_p|_\lambda}$ and $\varphi_p(1/\lambda) > \frac{1}{2}$ if and only if $\lambda < 2^{1/p}$, we obtain

COROLLARY 4. $l_{p,\lambda}$ *has normal structure if and only if* $\lambda < 2^{1/p}$ *and* $p > 1$.

The case $p = 2$ has been investigated directly in [**1**].

Theorem 3 shows that, among the Orlicz sequence spaces, normal structure and the sum-property are equivalent. The same holds for the related weak properties as Theorem 4 shows.

THEOREM 4. *The following are equivalent.*
(a) l_φ *has weakly normal structure.*
(b) l_φ *has the weak sum-property.*
(c) φ *satisfies the* Δ_2*-condition at* 0 *and* $C(\varphi) > \frac{1}{2}$ *or* $\varphi'(0) > 0$.

PROOF. (b) $\Rightarrow$ (a): Trivial. (c) $\Rightarrow$ (b): If $\varphi'(0) > 0$, then, by Theorem 2, every weakly convergent sequence converges in norm, i.e., $\lim_{n\to\infty}\Lambda(x_n) = 0$. So, l_φ has the weak sum-property. If φ satisfies the Δ_2-condition at 0 and $C(\varphi) > \frac{1}{2}$, then, by Proposition 6, (G_ε) holds for some $\varepsilon \in (0,1)$ and, hence, the basis $(e_i)_{i\in\mathbf{N}}$ of l_φ satisfies (GLD). According to Proposition 2, l_φ has the weak sum-property.

(a) $\Rightarrow$ (c): Assume that $\varphi'(0) = 0$, φ satisfies the Δ_2-condition at 0 and $C(\varphi) \leqslant \frac{1}{2}$. Then by Remark 1, $(e_n)_{n\in\mathbf{N}}$ is limit-constant, and, by Theorem 2, $(e_n)_{n\in\mathbf{N}}$ converges weakly to 0. So, l_φ does not have weakly normal structure. It remains to show that l_φ does not have weakly normal structure if φ does not satisfy the Δ_2-condition at 0. This is done in §5. □

COROLLARY 5. *If* φ *is real valued or if* $\frac{1}{2} < \varphi(t) < \infty$ *for at least one* t, *then* l_φ *has weakly normal structure if and only if either* l_φ *has no subspace isomorphic to* l_∞ *or* l_φ *is isomorphic to* l_1.

In [**10**, Theorem 1], Lami-Dozo shows that the condition

> (LD) Every bounded sequence in l_φ has a subsequence which is pointwise and almost convergent.

holds if φ is convex and satisfies

> (CV) For all $R > 0$ the map k_R: $(0,1) \to \mathbf{R}_+$ defined by $k_R(\gamma) = \inf_{0<t<R}\varphi(\gamma t)/\varphi(t)$ satisfies $0 < k_R(\gamma) < 1$ and $\lim_{\gamma\uparrow 1} k_R(\gamma) = 1$.

Clearly, (CV) is only well defined for real valued φ and implies the Δ_2-condition since $k_R(\gamma) = (\beta(\gamma^{-1}, \gamma R))^{-1}$. In [**10**], (CV) is only used to prove the following condition:

(*) For all $\varepsilon \in (0, \frac{1}{2})$, there is an $r > 0$ such that $\|x\|_\varphi \geqslant 1 + r$ whenever

$$\left\|\sum_{i=N+1}^{\infty} x(i)e_i\right\|_\varphi = 1 \quad \text{and} \quad \left\|\sum_{i=1}^{N} x(i)e_i\right\|_\varphi > \varepsilon \quad \text{for some } N \in \mathbf{N}.$$

which in turn implies (LD) even if l_φ is replaced by any normed space with boundedly complete basis $(e_i)_{i\in\mathbf{N}}$.

The proof of [**10**, Theorem 1] in fact only works for real valued φ. But for real valued φ, we even know from Proposition 4 that (CV) is equivalent to the Δ_2-condition. So, (LD) holds if φ satisfies the Δ_2-condition at 0 and is real valued. Moreover, in our general case, we even have a characterization of those Orlicz sequence spaces for which (LD) holds:

THEOREM 5. *Every bounded sequence in* l_φ *has a subsequence which is pointwise and almost convergent if and only if* φ *satisfies the* Δ_2*-condition at* 0 *and* $C(\varphi) = 1$.

PROOF. *Sufficiency.* If φ satisfies the Δ_2-condition at 0 and $C(\varphi) = 1$, then (G_ε) holds for all $\varepsilon \in (0,1)$. This implies $(*)$ and then (LD).

Necessity. If $C(\varphi) < 1$ and φ satisfies the Δ_2-condition at 0, then, according to Remark 1, $(e_n)_{n\in\mathbf{N}}$ has no subsequence which is almost convergent to its pointwise limit 0. Thus, (LD) does not hold. If φ does not satisfy the Δ_2-condition at 0, then l_φ does not have weakly normal structure. Hence, there is a sequence $(x_n)_{n\in\mathbf{N}}$ in l_φ which converges weakly to some $x \in l_\varphi$ such that $\Lambda(x) = \Lambda(x_n)$ for all $n \in \mathbf{N}$ (cf. [12]). So, (LD) cannot hold. □

4. Proof of Theorem 1.

(4.1) *Necessity.* Let φ satisfy the Δ_2-condition at 0 and be not linear at 0. It is enough to show that the following property holds:

(C) If $\|x_n\|_\varphi = 1$ and $x_n \to 0$ pointwise, then $\|x_1 + x_n\|_\varphi < 2$ for some $n > 1$.

Indeed, if $\|\Sigma_{k=1}^n a_k y_k\|_\varphi = \Sigma_{k=1}^n |a_k|$ for every finite sequence $(a_k)_{k\leqslant n}$ of scalars, then we may choose a subsequence $(z_n)_{n\in\mathbf{N}}$ of $(y_n)_{n\in\mathbf{N}}$ which converges pointwise and put $x_n = \frac{1}{2}(z_{2n+1} - z_{2n})$. Then, clearly, $x_n \to 0$ pointwise, $\|x_n\|_\varphi = 1$ and $\|x_1 + x_n\|_\varphi = 2$ for all $n \geqslant 1$.

We show (C). Assume $\|x_n\|_\varphi = 1$ and $x_n \to 0$ pointwise. Fix any $\gamma \in (1,2)$. Pick $\tilde{i} \in \mathbf{N}$ such that $|x_1(i)| < (\gamma - 1)t(\varphi)$ for all $i > \tilde{i}$. Choose $N \in \mathbf{N}$ such that $|x_n(i)| < (\gamma - 1)t(\varphi)$ for all $i \leqslant \tilde{i}$ and $n > N$. We have (i) $\|u_n\|_\infty < \gamma t(\varphi)/2$ for all $n > N$, $u_n := \frac{1}{2}(x_1 + x_n)$. Choose $j \in \mathbf{N}$ such that $t = |x_1(j)| > 0$. Then, $t \leqslant t(\varphi)$ and $0 < \mu = \lim_{s\uparrow t}\varphi(s) \leqslant 1$. Since φ is not linear at 0, there are $c > 1$ and $v > \frac{1}{2}t$ such that $\varphi(cv) = \frac{1}{2}\mu$. Since $\lim_{n\to\infty}|u_n(j)| = t/2$, there is $n > N$ such that $|u_n(j)| < v$. So, $\varphi(\alpha|u_n(j)|) < \alpha c^{-1}\mu/2$ for all $\alpha \in (1, c]$. If $1 < \alpha < \min\{\sqrt{2/\gamma}, c\}$ then

$$S_\varphi(\alpha u_n) < \beta\Big(\alpha^2, \frac{\gamma}{2}t(\varphi)\Big)\sum_{i\neq j}\varphi\big(\alpha^{-1}|u_n(i)|\big) + \alpha c^{-1}\mu/2$$

$$\leqslant \beta\Big(\alpha^2, \frac{\gamma}{2}t(\varphi)\Big)\sum_{i\neq j}\frac{1}{2}\big\{\varphi\big(\alpha^{-1}|x_1(i)|\big) + \varphi\big(\alpha^{-1}|x_n(i)|\big)\big\} + \alpha c^{-1}\mu/2$$

$$< \beta\Big(\alpha^2, \frac{\gamma}{2}t(\varphi)\Big)\Big(1 - \frac{1}{2}\varphi(\alpha^{-1}t)\Big) + \alpha c^{-1}\mu/2 \to 1 - (1 - c^{-1})\mu/2 < 1 \quad \text{if } \alpha \downarrow 1.$$

Hence, $S_\varphi(\alpha u_n) < 1$ for some $\alpha > 1$ and, consequently, $\|x_1 + x_n\|_\varphi \leqslant 2\alpha^{-1} < 2$.

(4.2) *Sufficiency.* According to Proposition 7, it remains to show that l_φ has a subspace isometric to l_1 if φ does not satisfy the Δ_2-condition at 0. So, assume that φ does not satisfy the Δ_2-condition at 0. It is enough to find a sequence $(x_n)_{n\in\mathbf{N}}$ in l_φ such that $\|x_n\|_\varphi \leqslant 1$ for all $n \in \mathbf{N}$ and $\|\Sigma_{k=1}^n a_k x_k\|_\varphi \geqslant \Sigma_{k=1}^n |a_k|$ for every finite sequence $(a_k)_{k\leqslant n}$ of scalars.

Assume first that there are sequences $(t_m)_{m\in\mathbf{N}} \subset \mathbf{R}_+$ and $(k_m)_{m\in\mathbf{N}} \subset \mathbf{N}$ such that

$$k_m\varphi(t_m) \leqslant 2^{-m}, \tag{1}$$

$$\liminf_{m\to\infty} k_m\varphi(\alpha t_m) > 0 \quad \text{for all } \alpha > 1. \tag{2}$$

Set $s_m = \Sigma_{j=1}^{m-1} k_j$ and $x_n(s_m + k) = \sigma_{n,m} t_m$, $1 \leqslant k \leqslant k_m$, where $\sigma_{n,m}$ are certain scalars with $|\sigma_{n,m}| = 1$ (to be defined below).

No matter how the $\sigma_{n,m}$ are defined, we have by (1) for all $n \in \mathbf{N}$:

$$S_\varphi(x_n) = \sum_{m=1}^{\infty} k_m \varphi(t_m) \leqslant \sum_{m=1}^{\infty} 2^{-m} = 1,$$

hence $\|x_n\|_\varphi \leqslant 1$.

We now define the $\sigma_{n,m}$. Set $r_j = j$ if $\mathbf{K} = \mathbf{C}$ and $r_j = 2$ if $\mathbf{K} = \mathbf{R}, j \in \mathbf{N}$. Let, for all $p \in \mathbf{N}$, $\Omega_p = (\omega_{p,n,\nu})$ be some $p \times r_p^p$ matrix whose columns are just all combinations of p-tuples of unit roots of order r_p. For example $\Omega_2 = \left(\begin{smallmatrix} +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \end{smallmatrix}\right)$. We define $(\sigma_{n,m}) = \left(\begin{smallmatrix} \Omega_1 \Omega_2 \Omega_3 \cdots \\ 1 \end{smallmatrix}\right)$ i.e.,

$$\sigma_{n,m} = \begin{Bmatrix} \omega_{p,n,\nu} & n \leqslant p \\ 1 & n > p \end{Bmatrix}, \quad m = \sum_{j=1}^{p-1} r_j^j + \nu, \qquad 1 \leqslant \nu \leqslant r_p^p.$$

Let scalars a_j, $j \leqslant n$, be given with $\Sigma_{j=1}^n |a_j| = 1$. We have to show that $\|\Sigma_{j=1}^n a_j x_j\|_\varphi \geqslant 1$. It is enough to show that $S_\varphi(\beta \Sigma_{j=1}^n a_j x_j) = \infty$ for all $\beta > 1$. So, fix $\beta > 1$. Pick $\gamma < 1$ such that $\alpha = \beta\gamma > 1$. Take $\tilde{p} > n$ such that, for every $p \geqslant \tilde{p}$ and $j \leqslant n$, there is a unit root $\zeta_{p,j}$ of order r_p such that

$$\text{(3)} \qquad \operatorname{Re}(a_j \zeta_{p,j}) \geqslant \gamma |a_j|, \ j \leqslant n, p \geqslant \tilde{p}, \text{ and, hence, } \left| \sum_{j=1}^n a_j \zeta_{p,j} \right| \geqslant \gamma, p \geqslant \tilde{p}.$$

By definition of Ω_p, there are numbers $m_p = \Sigma_{i=1}^{p-1} r_i^i + \nu_p$, $1 \leqslant \nu_p \leqslant r_p^p$, such that

$$\text{(4)} \qquad \zeta_{p,j} = \omega_{p,j,\nu_p} = \sigma_{j,m_p} \quad \text{for all } j \leqslant n, p \geqslant \tilde{p}.$$

Using (3) and (4), we finally obtain

$$S_\varphi\left(\beta \sum_{j=1}^n a_j x_j\right) \geqslant \sum_{p=\tilde{p}}^{\infty} k_{m_p} \varphi\left(\beta t_{m_p} \left| \sum_{j=1}^n a_j \zeta_{p,j} \right| \right) \geqslant \sum_{p=\tilde{p}}^{\infty} k_{m_p} \varphi(\alpha t_{m_p}) = \infty$$

because, by (2), $\liminf_{p \to \infty} k_{m_p} \varphi(\alpha t_{m_p}) > 0$.

It remains to construct the sequences $(t_m)_{m \in \mathbf{N}}$ and $(k_m)_{m \in \mathbf{N}}$ with (1) and (2). We distinguish two cases: (a) φ is degenerate. Then $t = \sup\{s \geqslant 0 | \varphi(s) = 0\}$. Set $k_m = 1$ and choose an arbitrary sequence $(t_m)_{m \in \mathbf{N}}$ with $t_m \uparrow t$. Then, $k_m \varphi(t_m) = 0 < 2^{-m}$. Given $\alpha > 1$, there are $u > t$ and $\tilde{m} \in \mathbf{N}$ such that $\alpha t_m > u$ for all $m > \tilde{m}$. Hence, $\liminf_{m \to \infty} k_m \varphi(\alpha t_m) \geqslant \varphi(u) > 0$.

(b) φ is nondegenerate. Then, there is a sequence $(t_m)_{m \in \mathbf{N}}$ with $\varphi(t_m) \leqslant 2^{-m-1}$ and $\varphi(\alpha t_m)/\varphi(t_m) \geqslant (\alpha - 1)2^m$ for all $\alpha \in (1,2)$ (Proposition 4(ii)). Let k_m be the largest integer with $2^m k_m \varphi(t_m) \leqslant 1$. For all $\alpha \in (1,2)$: $k_m \varphi(\alpha t_m) \geqslant (\alpha - 1)2^m \varphi(t_m)$ $= (\alpha - 1)(2^m(k_m + 1)\varphi(t_m) - 2^m \varphi(t_m)) \geqslant (\alpha - 1)/2 > 0$. □

5. Remaining part of the proof of Theorem 3. We assume that $C(\varphi) > \frac{1}{2}$, φ satisfies the Δ_2-condition at 0 and φ is not linear at 0. Suppose that l_φ does not have the sum-property. Then, there is a limit-affine sequence $(x_n)_{n \in \mathbf{N}}$ in l_φ with $\inf_{n \in \mathbf{N}} \Lambda(x_n) \geqslant \varepsilon > 0$. If $r > 0$, $x \in l_\varphi$ and $(y_n)_{n \in \mathbf{N}}$ is a subsequence of $(x_n)_{n \in \mathbf{N}}$, then $(r(y_n - x))_{n\in\mathbf{N}}$ is also limit-affine with $\inf_{n\in\mathbf{N}}\Lambda(r(y_n - x)) \geqslant r\varepsilon$. Since pointwise limits of bounded sequences in l_φ belong to l_φ—the basis is boundedly complete—we may assume that $x_n \to 0$ pointwise and $\Lambda(x_n) \to 1$.

Since $C(\varphi) > \frac{1}{2}$, there is $\tilde{\beta} < 1$ such that $2\varphi(\tilde{\beta}t) > 1$, $t = t(\varphi)$. We choose an arbitrary $\hat{\beta} \in (\tilde{\beta}, 1)$ and claim:

(1) There exists $N \in \mathbf{N}$ such that $\|x_n\|_\infty \leqslant \hat{\beta}t$ for all $n > N$.

We proof (1) indirectly. If (1) does not hold, then—since $x_n \to 0$ pointwise—for all $\tilde{i} \in \mathbf{N}$ and $N \in \mathbf{N}$, there are $i \geqslant \tilde{i}$ and $n > N$ such that $|x_n(i)| > \hat{\beta}t$. Pick $\alpha > 1$ with $\alpha\tilde{\beta} < \hat{\beta}$ and put $\varepsilon = (\hat{\beta} - \alpha\tilde{\beta})t > 0$. Choose $M \in \mathbf{N}$ such that $\Lambda(x_n) < \alpha$ for all $n > M$. There are $k > M$ and $i \geqslant 1$ such that $|x_k(i)| > \hat{\beta}t$. Take $\tilde{i} > i$ such that $|x_k(j)| < \varepsilon$ for all $j \geqslant \tilde{i}$. Pick $N > k$ such that $|x_n(i)| < \varepsilon$ for all $n > N$ and (i): $\|x_n - x_k\|_\varphi < \alpha$ for all $n > N$. There are $n > N$ and $j \geqslant \tilde{i}$ such that $|x_n(j)| > \hat{\beta}t$. This all together yields $|x_n(i) - x_k(i)| > \hat{\beta}t - \varepsilon = \alpha\tilde{\beta}t$, $|x_n(j) - x_k(j)| > \hat{\beta}t - \varepsilon = \alpha\tilde{\beta}t$. So, $S_\varphi(\alpha^{-1}(x_n - x_k)) > 2\varphi(\tilde{\beta}t) > 1$ which contradicts (i) and completes the proof of (1).

Fix any $\beta \in (\hat{\beta}, 1)$ and set $\delta = (\beta - \hat{\beta})t$. Using $\lim_{n\to\infty} x_n(i) = 0$, $i \in \mathbf{N}$, and $\lim_{i\to\infty} x_n(i) = 0$, $n \in \mathbf{N}$, we construct inductively a subsequence of $(x_n)_{n\in\mathbf{N}}$—again denoted by $(x_n)_{n\in\mathbf{N}}$—and an increasing sequence $(i_n)_{n\in\mathbf{N}}$ of indices such that

(2) $$|x_n(i)| < \delta \quad \text{whenever } i \leqslant i_{n-1} \text{ or } i \geqslant i_n,\ n \in \mathbf{N}.$$

Using the triangle inequality pointwise, we obtain $\|x_m - x_n\|_\infty < \hat{\beta}t + \delta = \beta t$.

Since $\Lambda(x_n) \to 1$, there is a subsequence of $(x_n)_{n\in\mathbf{N}}$—again denoted by $(x_n)_{n\in\mathbf{N}}$—and $\gamma < 1$ such that $\|x_m - x_n\|_\varphi > \gamma^{-1}\beta$ for all $n \neq m$. Therefore

(3) $$\|x_m - x_n\|_\infty < \beta t < \gamma t\|x_m - x_n\|_\varphi \quad \text{for all } m \neq n.$$

Choose $j \in \mathbf{N}$ such that $s = |x_1(j)|/\Lambda(x_1) > 0$. We have $s \leqslant \gamma t$. Set $\lambda = \Lambda(x_1)/(1 + \Lambda(x_1))$. Since φ is continuous on $[0, \gamma t]$ and not linear at 0, there are $c > 1$ and $u > \lambda s$ such that $\varphi(cu) = \lambda\varphi(s)$. Hence,

$$\lim_{m\to\infty}\lim_{n\to\infty}|y_{n,m}(j)| = \lambda s, \quad y_{n,m} := (2x_n - x_1 - x_m)/\left(\|x_n - x_1\|_\varphi + \|x_n - x_m\|_\varphi\right),$$

implies

(4) $$\lim_{m\to\infty}\lim_{n\to\infty}\varphi\left(\alpha|y_{n,m}(j)|\right) \leqslant \alpha c^{-1}\lambda\varphi(s) \quad \text{for all } \alpha \in (1, c).$$

For $\lambda_{n,m} = \|x_n - x_1\|_\varphi/(\|x_n - x_1\|_\varphi + \|x_n - x_m\|_\varphi)$, $z_{n,l} = (x_n - x_l)/\|x_n - x_l\|_\varphi$ we have

(5) $$y_{n,m} = \lambda_{n,m}z_{n,1} + (1 - \lambda_{n,m})z_{n,m}, \qquad n > m > 1,$$

(6) $$\|z_{n,l}\|_\infty < \gamma t \text{ and } S_\varphi(z_{n,l}) \leqslant 1, \qquad n > l \geqslant 1.$$

Set $\tilde{\alpha} = \min(c, \gamma^{-1}) > 1$. Application of (5) and (6) yields for $\alpha \in (1, \tilde{\alpha})$:

(7) $$\begin{aligned} S_\varphi(\alpha y_{n,m}) &\leqslant \beta(\alpha, \gamma t)\sum_{i\neq j}\left\{\lambda_{n,m}\varphi\left(|z_{n,1}(i)|\right) + (1 - \lambda_{n,m})\varphi\left(|z_{n,m}(i)|\right)\right\} \\ &\quad + \varphi\left(\alpha|y_{n,m}(j)|\right) \\ &\leqslant \beta(\alpha, \gamma t)\left\{1 - \lambda_{n,m}\varphi\left(|z_{n,1}(j)|\right)\right\} + \varphi\left(\alpha|y_{n,m}(u)|\right) \\ &:= h(\alpha, n, m). \end{aligned}$$

Using (4), $\lim_{\alpha\downarrow 1}\beta(\alpha,\gamma t)=1$, $\lim_{m\to\infty}\lim_{n\to\infty}\lambda_{n,m}=\lambda$ and $\lim_{n\to\infty}|z_{n,1}(j)|=s$, we deduce $\lim_{\alpha\downarrow 1}\lim_{m\to\infty}\lim_{n\to\infty}h(\alpha,n,m)=1-\lambda(1-c^{-1})\varphi(s)<1$.

So, by (7), there is $\alpha\in(1,\tilde{\alpha})$, $m>1$ and $N>m$ such that $S_\varphi(\alpha y_{n,m})\leqslant h(\alpha,n,m)<1$ for all $n>N$. From the very definition of $y_{n,m}$ we finally obtain $\|x_n-\frac{1}{2}(x_1+x_m)\|_\varphi\leqslant\alpha^{-1}\frac{1}{2}(\|x_n-x_1\|_\varphi+\|x_n-x_m\|_\varphi)$, $n>N$. Letting $n\to\infty$, this contradicts the limit-affineness of $(x_n)_{n\in\mathbf{N}}$. □

6. Remaining part of the proof of Theorem 4. We suppose that φ does not satisfy the Δ_2-condition at 0. Set $t=\max\{s>0|\varphi(s)=0\}$. We distinguish the two cases $t=0$ and $t>0$. In each case, we construct sequences $(t_n)_{n\in\mathbf{N}}$ and $(\alpha_n)_{n\in\mathbf{N}}$ in $\mathbf{R}_+$ and $(k_n)_{n\in\mathbf{N}}$ in $\mathbf{N}$ with $\alpha_n\downarrow 1$, $k_n\varphi(t_n)<2^{-n}$ and $k_n\varphi(\alpha_n t_n)\geqslant 1$.

Case 1. $t>0$. Define $\alpha_n>1$ by $\varphi(\alpha_n t)=(kn)^{-1}$, $k>C(\varphi)^{-1}$. Set $k_n=kn$ and $t_n=t$. Then $k_n\varphi(\alpha_n t_n)=1$ and $k_n\varphi(t_n)=0<2^{-n}$.

Case 2. $t=0$. Using Proposition 4(ii), we inductively construct a decreasing sequence $(t_n)_{n\in\mathbf{N}}$ of positive numbers with

(i) $\varphi(\alpha_n t_n)>2^{n+1}\varphi(t_n)$, $\alpha_n:=1+n^{-1}$, $\varphi(2t_1)\leqslant 1$.

Let k_n be the least integer with $k_n\varphi(\alpha_n t_n)\geqslant 1$. We have

$$k_n\varphi(\alpha_n t_n)\leqslant(k_n-1)\varphi(\alpha_n t_n)+\varphi(2t_1)<1+1=2$$

and, thus, $k_n\varphi(t_n)<2^{-n-1}k_n\varphi(\alpha_n t_n)<2^{-n}$.

Both cases. We set $s_n=\sum_{i=1}^{n-1}k_i$ and $x_n=t_n\sum_{i=1}^{k_n}e_{s_n+i}\in l_\varphi$. The x_n have mutually disjoint supports. So, for arbitrary $\xi=(\xi_n)_{n\in\mathbf{N}}\in c_0$, the pointwise sum $T\xi=\sum_{n=1}^{\infty}\xi_n x_n$ is well defined. Since

$$S_\varphi\left(\|\xi\|_\infty^{-1}T\xi\right)\leqslant\sum_{n=1}^{\infty}k_n\varphi(t_n)<\sum_{n=1}^{\infty}2^{-n}=1,$$

$T\colon c_0\to l_\varphi$ is linear and bounded with $\|T\|\leqslant 1$. Obviously, $x_n=T\delta_n$, δ_n the nth unit vector of c_0. Since $\delta_n\to 0$ weakly in c_0, $x_n\to 0$ weakly in l_φ.

Let $x=\sum_{k=1}^{n}\lambda_k x_k$, $\lambda_k\geqslant 0$, $\sum_{k=1}^{n}\lambda_k=1$, be arbitrarily given. Then, $\|x_n-x\|_\varphi\leqslant\|\delta_n-\sum_{k=1}^{n}\lambda_k\delta_k\|_\infty=1$, $n>m$, and so $\Lambda^*(x)\leqslant 1$.

Conversely, we have $\Lambda_*(x)\geqslant 1$ since $\|x_n-x\|_\varphi\geqslant\alpha_n^{-1}$, $n>m$, because of $S_\varphi(\alpha_n(x_n-x))\geqslant k_n\varphi(\alpha_n t_n)\geqslant 1$, $n>m$.

Hence, $\Lambda^*(x)=\Lambda_*(x)=1$ and $(x_n)_{n\in\mathbf{N}}$ is limit-constant. Therefore, l_φ does not have weakly normal structure. □

ADDED IN PROOF. The author has shown the sum-property does not imply normal structure (cf. the text after Proposition 1) and that an Orlicz sequence space has a subspace isomorphic to l_∞ if and only if it has a subspace isometric to l_∞ (cf. the text after Theorem 1). These results will be published elsewhere.

REFERENCES

1. J. P. Baillon and R. Schöneberg, *Asymptotic normal structure and fixed points of nonexpansive maps*, Proc. Amer. Math. Soc. **81** (1981), 257–264.

2. M. S. Brodskii and D. P. Mil'man, *On the center of a convex set*, Dokl. Akad. Nauk SSSR **59** (1948), 837–840. (Russian)

3. F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 875–881.

4. M. Edelstein, *The construction of an asymptotic center with a fixed point property*, Bull. Amer. Math. Soc. **78** (1972), 206–208.

5. K. Goebel, *On a fixed point theorem for multivalued nonexpansive mappings*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **29** (1975), 69–72.

6. D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr. **30** (1965), 251–258.

7. J. P. Gossez and E. Lami-Dozo, *Structure normale et base de Schauder*, Bull. Acad. Royale Belgique **15** (1969), 673–681.

8. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006.

9. E. Lami-Dozo, *Centres asymptotiques dans certains F-espaces*, Boll. Un. Mat. Ital. B **17** (1980), 740–747.

10. ______, *Asymptotic centers in particular spaces*, Fixed Point Theory (Proc. Conf. Sherbrooke, 1980), Lecture Notes in Math., vol. 886, Springer-Verlag, Berlin and New York, 1981.

11. Th. Landes, *A characterization of totally normal structure*, Arch. Math. **37** (1981), 248–255.

12. ______, *Permanence properties of normal structure*, Pacific J. Math. **110** (1984), 125–143.

13. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*. I, *Sequence spaces*, Springer-Verlag, Berlin and New York, 1977.

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